THREE DIMENSIONAL MATHEMATICAL MODEL OF TIME –DEPENDENT CONVECTION IN SOLAR ENERGY SYSTEMS

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ABSTRACT

This paper presents an overview of the main aspects involved in the mathematical model of convection of fluids in three-dimensional space with time varying boundary conditions. This model considers the solar radiation in the boundary conditions and their effects on the fluid flow.

1. INTRODUCTION

The growing world energy consumption, the limited amount of existing fossil fuel, and the contamination of the environment let us know the necessity of using efficient energy transformation systems and, whenever it is possible, renewable sources of energy. Consequently, it is important to understand the relation among all the variables related to energy systems to analyze the feasibility of adopting the use of renewable sources of energy, and it is also important to predict how efficiently a new energy system design would work. In both cases, a model which is a mathematical representation of a device or process is a useful tool that can be used to understand and predict the behavior of any energy system, and to optimize its efficiency.

The convection of fluids and the use of solar radiation are important aspects of the energy systems, as well as systems that take advantage of thermal energy transmitted by the solar radiation. The objective of this work is to present an overview of the main aspects for developing mathematical models of a number of energy systems. The description will refer to energy systems that can be represented by figure 1 and will be applicable to describe energy systems where natural or forced convection of fluids, and solar radiation play important roles. Figure 1 represents a generalized geometry of energy generation systems that use solar radiation and fluid flows to transform the energy through an internal body Bin. The model can be used, for example, for the analysis of turbines moved by fluid convection, the effect of solar radiation and fluid convection around photovoltaic panels, green buildings, among others.

2. FLUID FLOW MODEL

2.1 Mathematical model

The mathematical model consist of a set of partial differential equations governing the flow of a fluid in a three dimensional geometry limited by an external and an internal boundary (Bout and Bin in figure 1). Inlets and outlets of the fluid, represented by the walls W_1 and W_2 respectively, are considered over the outer boundary, as well as semitransparent windows, W_i, through which solar radiation enters to the system. The inner boundary, Bin, is considered a boundary of a subsystem to which energy from the fluid flow and/or the solar radiation is transferred. The approach used in the model is considered general enough to get a realistic description and prediction of the performance of a wide variety of energy systems.

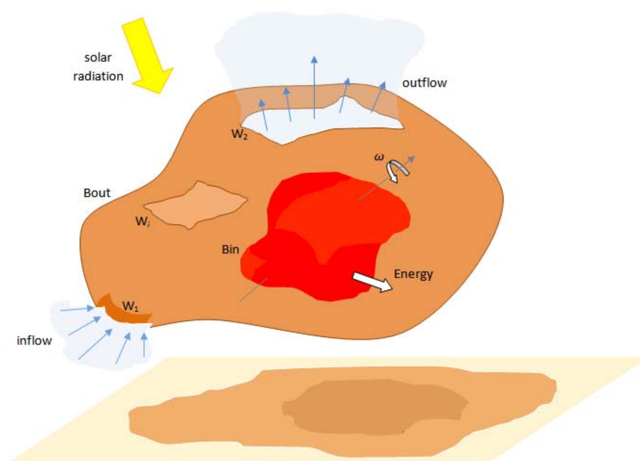

Fig. 1: Three dimensional representation of energy systems that use solar radiation.

The present modeling of natural convection in fluids is built on the following assumptions:

- The fluid is continuous, homogeneous and Newtonian.
- The fluid is in chemical equilibrium.

With the assumptions mentioned above, the convection of a fluid in a three-dimensional space and its evolution over time is described by five independent variables. With conservation laws, the governing equations take the form of the following vector equation (Jansen, Whiting, & Hulbert, 2000; Patankar, 1980)

$$\frac{\partial \mathbf{U}}{\partial \mathrm{t}} + \nabla.\mathrm{F}^{\mathrm{adv}} - \nabla.\mathbf{F}^{\mathrm{diff}} - \mathbf{F}^{\mathrm{rss}} = \mathbf{0}, \qquad (2.1)$$

where U is the vector of conservation variables,

$$\mathbf{F}^{\mathrm{adv}} = [\mathrm{F}_1^{\mathrm{adv}}\ \mathrm{F}_2^{\mathrm{adv}}\ \mathrm{F}_3^{\mathrm{adv}}]^{\dagger} \qquad (2.2)$$

$\mathbf{F}_{\mathbf{i}}^{\mathbf{adv}}$ is a vector that contains the advective fluxes in the ith-direction,

$$\boldsymbol{F}^{\mathrm{diff}} = [F_1^{\mathrm{diff}}\ F_2^{\mathrm{diff}}\ F_3^{\mathrm{diff}}]^{\dagger} \qquad (2.3)$$

$\boldsymbol{F}_i^{diff}$ is a vector that contains the diffusive fluxes in the ith-direction, † is the transpose operator, and $\boldsymbol{F}_i^{rss}$ is a vector that contains the reactive, source and sink terms.
With the chain rule, it is possible to rewrite equation (2.1) in a quasi-linear form using any set of independent variables **Y** and get equation (2.4).

$$R(Y) = Tr + Ad - Di - RSS = 0, \qquad (2.4)$$

where $Tr = \mathbf{A}_0 \frac{\partial \mathrm{Y}}{\partial t}$, $Ad = (\mathbf{A}.\nabla)\mathbf{Y}$, $Di = \nabla.(\mathbf{K}\nabla\mathbf{Y})$, and $RSS = \boldsymbol{S_1 Y} + \boldsymbol{S_0}$, R is the vector of residuals, $\mathbf{A_0}$ is the change of variables metric, **A** is a matrix composed by Euler Jacobian matrixes, **K** is a matrix composed by diffusivity matrixes, $\mathbf{S_1}$ and $\mathbf{S_0}$ are vectors used to represent the reactive, source, and sink terms.

The particular set of variables **Y** taken in this work is

$$Y = [p\ u_1\ u_2\ u_3\ \ T]^{\dagger} \qquad (2.5)$$

where p is pressure, u_i is the Cartesian velocity component in the ith-direction and T is the absolute temperature.
Equation (2.1) and consequently equation (2.4) are based on the equations of conservation of mass, conservation of momentum and conservation of energy. These equations are of the form Transient + Advective – Diffusive – Reactive/Source/Sink = 0 and their terms are shown in table 1.

The terms of the conservation of energy equation, in table 1, include conduction, convection, and contribution by radiative heat transfer, as well as internal heat sources, compression work, viscous dissipation, and energy storage due to transients. In table 1 ρ is the mass density, t time, $\overleftrightarrow{\tau}$ is the stress tensor, $\vec{f}$ the body force per unit volume, h enthalpy, $\overleftrightarrow{q'}$ is the total heat flux, q''' the local heat source per unit volume and time, and Φ_d is the heat production by viscous dissipation.

TABLE 1: EQUATION TERMS OF THE FLUID FLOW MODEL

Equation	Transient	Advective	Diffusive	Reactive/ Source/ Sink
Conservation of mass	$\frac{\partial \rho}{\partial t}$	$\vec{u}.\nabla\rho + \rho\nabla.\vec{u}$	0	0
Conservation of momentum	$\rho\frac{\partial \vec{u}}{\partial t}$	$\rho\vec{u}.\nabla\vec{u} + \nabla p$	$-\nabla\overleftrightarrow{\tau}$	$\vec{\mathrm{f}}$
Conservation of energy	$\rho\frac{\partial h}{\partial t} - \frac{\partial p}{\partial t}$	$\rho\vec{u}.\nabla h - \vec{u}.\nabla p$	$-\nabla\overleftrightarrow{q'}$	$-\mathrm{q}''' - \Phi_{\mathrm{d}}$

2.2 Diffusive terms

The terms containing the diffusive fluxes in table 1 are calculated considering the following relations (Bird, Steward, & Lightfoot, 2002; Jaluria & Torrance, 2003):

$$\overleftrightarrow{\tau} = -\mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})') + \left(\frac{2}{3}\mu - \kappa\right)(\nabla.\boldsymbol{u})\overleftrightarrow{\boldsymbol{\delta}} \qquad (2.6)$$

$$\overrightarrow{q'} = -k\nabla T + \overrightarrow{q_r'} \qquad (2.7)$$

Where μis the fluid viscosity, κ is called the dilatational viscosity, $\overleftrightarrow{\delta}$ is the unit tensor, k the thermal conductivity, and $\overleftrightarrow{q'}$ the radiation contribution to the total heat flux.
The radiative flux, $\overrightarrow{q_r'}$, is present in expression (2.7), to account for the net thermal radiation from the sun gained by a volume element, because the working fluid (as water

vapor, carbon dioxide, gases associated with combustion chambers at high pressure and temperature, plasma, among others) is considered an absorbing, scattering and emitting medium. $\vec{q_r'}$, is considered negligible when the radiant energy density is much smaller than the total energy of the fluid, the fluid velocity is much smaller than the velocity of the light, and when the radiation pressure is much smaller than the pressure of the fluid (Viskanta, 1998). The mathematical analysis of this part of this work is challenging mainly due to two reasons. First, the knowledge of the temperature, radiation intensity, and physical properties throughout the medium is required, because the absorption, emission, and scattering of energy occur not only at the boundaries of the system but at all locations in it. Other difficulty is that a detailed spectrally dependent analysis may be required, because the spectral effects are often more pronounced in gases than in solids boundaries.

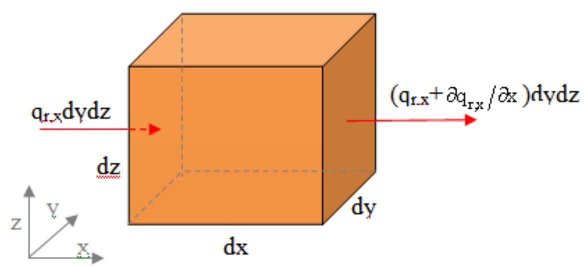

Fig. 2: Radiative heat flux acting on a volume element of a participating medium.

Figure 2 shows the radiative energies in and out of the dy dz faces of a volume element dxdydz; with the inclusion of the analysis of the radiative energies corresponding to faces dx dz and dx dy, the subtraction of the outgoing energies from the incoming, and the division of the result by dxdydz, the radiative energy supplied per unit volume is found:

$$-\left[\frac{\partial q_{r,x}}{\partial x}+\frac{\partial q_{r,y}}{\partial x}+\frac{\partial q_{r,z}}{\partial x}\right]=-\nabla.\vec{q_r} \qquad (2.8)$$

which is related with the radiative heat flux $\vec{q_r'}$. To obtain the components $q_{r,x}$, $q_{r,y}$ and $q_{r,z}$ of the radiative heat flux, or in general a component $q_{r,n}$ in a direction $\vec{n}$ crossing an area dA normal to the unit vector $\vec{n}$, it is necessary to integrate the contributions by the local intensities crossing in all directions and at all wavelengths, which is expressed as

$$q_{r,n}=\int_{\omega=0}^{4\pi} i'\,\vec{s}.\vec{n}d\omega, \qquad (2.9)$$

where i' is the total intensity on the direction $\vec{s}$ and ω a solid angle.

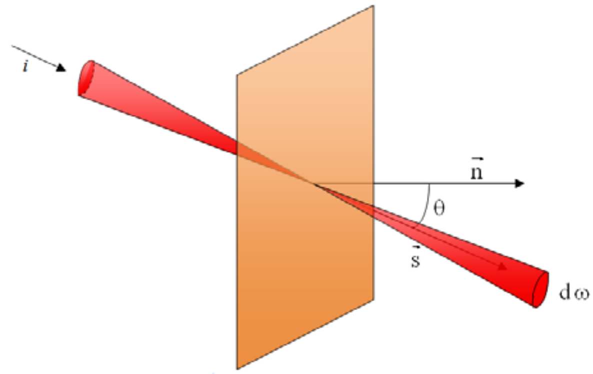

Fig. 3: Total intensity i incident from a direction $\vec{s}$ on an area dA.

The intensity can be called *spectral intensity* when it refers to radiation in an interval $d\lambda$ around a single wavelength, or *total intensity* when it refers to combined radiation including all wavelengths; the total intensity is calculated integrating the spectral intensity over all wavelengths.

$$i'=\int_{\lambda=0}^{\infty} i'_\lambda d\lambda, \qquad (2.10)$$

In equation (2.10) i is intensity, the superscript prime, and the subscript λ are used to denote, respectively, that radiation per unit solid angle in a single direction, and that a spectral quantity are being considered.

The local radiation traveling in a single direction per unit solid angle and wavelength, crossing a unit area, i'_λ, is given by the radiative transfer equation, which describes the radiation intensity at any position along a path through an absorbing, emitting and scattering medium (Siegel & Howell, 2002; Watson & Chapman, 2002).

$$\frac{di'_\lambda}{ds}=-a_\lambda(\vec{s})i'_\lambda(\vec{s})+a_\lambda(\vec{s})i'_{\lambda b}(\vec{s})-\sigma_{s\lambda}(\vec{s})i'_\lambda(\vec{s})+\frac{\sigma_{s\lambda}(\vec{s})}{4\pi}\int_{\omega_i=0}^{4\pi} i'_\lambda(\vec{s},\omega_i)\,\Phi_\lambda(\omega,\omega_i)d\omega_i \qquad (2.11)$$

Where a_λ is the absorption coefficient, $\sigma_{s\lambda}$ the scattering coefficient; $K_\lambda=a_\lambda+\sigma_{s\lambda}$ is known as the extinction coefficient, and Φ_λ is the phase function for scattering.

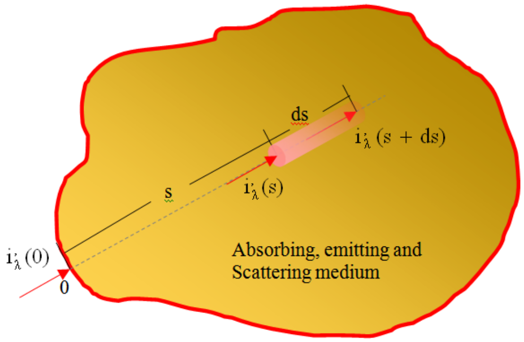

Fig. 4: Coordinates of the radiative transfer equation.

2.3 Reactive, source, and sink terms

The body force per unit volume, $\vec{f}$, is produced by a field, which can be the gravitational, electromagnetic, etc. For the case in which the only field affecting the unit volume is gravity $\vec{f}$ is $\rho\vec{g}$.

On the modeling of the fluid convection of the energy system analyzed, it is considered that there is no local heat source per unit volume and time (i.e., $q''' = 0$).

For Cartesian coordinates, Φ_d, the heat production by viscous dissipation, is calculated using the following expressions (Bird, Steward, & Lightfoot, 2002) :

$$\Phi_d = \mu\Phi_v \qquad (2.12)$$

$$\Phi_v = 2\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] + \left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]^2 + \left[\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right]^2 + \left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right]^2 - \frac{2}{3}\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right]^2 \qquad (2.13)$$

This term is commonly neglected, because it is important only in fluid flows with very large velocity gradients.

2.4 Thermodynamic and transport properties of the fluid

In order to have a solvable set of differential equations it is necessary to have the properties of the fluid as a function of the variables chosen in our system of equations. The initial values of the thermodynamic and transport properties are expected to be known or easy to calculate as a function of pressure and temperature. As time increases the new values of the properties can be calculated as a function of pressure and temperature using the following differentiation relation:

$$\partial h = \frac{\partial h}{\partial p}\partial p + \frac{\partial h}{\partial T}\partial T \qquad (2.14)$$

where h represents a property of the fluid at certain instant; and p and T are the corresponding pressure and temperature of the fluid. Consequently the accuracy of the solution will depend, partially, on the accuracy of the relations used to calculate the variation of the property respect to pressure and temperature.

3. BOUNDARY CONDITIONS

In order to solve the system of equations that govern the fluid flow, it is necessary to know, besides the initial conditions of the set of variables or its derivatives, the boundary conditions over the spatial domain for the period of time that the system is analyzed.

To set the boundary conditions, the following considerations have to be taken into account: at solid-fluid interfaces the no-slip condition is satisfied (i.e., the fluid velocity equals the velocity with which the solid surface is moving), as well as the continuity of temperature and heat flux normal to the interface; far from the inlet and outlet, the temperature and pressure are from the surroundings, and the velocities are negligible when they are compared with the velocities at the inlet and outlet.

3.1 Interior boundary conditions

To set the interior boundary conditions the criteria mentioned above should be used, but also it is necessary to couple our initial system of equations, represented by equation (2.1), with the governing equations of the interior body (Bin in figure 1), which can be a turbine, photovoltaic panel, or any other system that is generating energy from the interaction with its surroundings.

3.2 Exterior boundary conditions

As seen in figure 1 the external boundary conditions are formed by inflow, outflow and wall boundary conditions.

If the boundary conditions at the inlet(s) of our control volume are not known, it is recommended to include a space surrounding the inlet of the initial chosen control volume into the computational domain, to set as boundary conditions the conditions of the surroundings, which are often constant.

For the boundary conditions at the outlet, although there will be an increment on the computational cost, it is recommended to add the space surrounding the original outlet(s) to the initial computational domain. The new outlet(s) should be far enough from the original outlet(s) to consider the velocities negligible, and pressure and temperature equal to the ones from the surroundings, on this way one gets more realistic results than the ones obtained by making assumptions about the parameters at the original outlet(s).

For the transparent walls, w_i, the main consideration is the time varying conditions of the heat flux due to the time-dependent solar radiation over the walls (Duffie & Beckman, 2006) .

4. AN APPLICATION EXAMPLE

A particular case illustrating the approach presented was the numerical model implemented for the analysis of a solar updraft tower. In this system the solar radiation heats the air below the collector roof, creating a gradient of temperature in the chimney, placed in the middle of the collector roof. It is the difference of temperature between the top and bottom zones of the chimney that creates the air flow which moves the turbine that generates electricity.

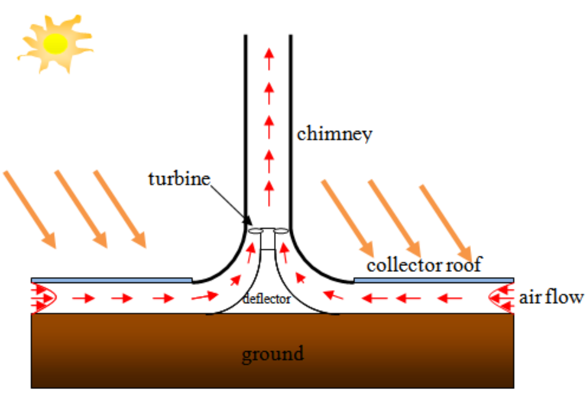

Fig. 5: Schematic representation of the air flow moving a turbine in a solar updraft tower.

The finite volume method based on the k-epsilon approach to describe a turbulent fluid was used with the commercial software Fluent 6.3; it also has the following advantages: it provides radiation models to deal with the radiative transfer equation, and a solar load model that can be used to include the sun’s radiation effects on the computational domain.

From the numerical analysis of 1 hour of a system installed in Manzanares, Spain, an average value for the velocity at the top of the internal deflector was about 14m/s.

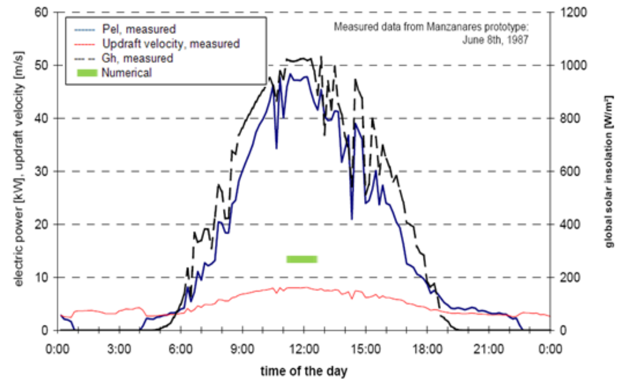

Fig. 6: Comparison of numerical simulation and experimental results (Schlaich, Bergermann, Schiel, & Weinrebe, 2004).

The higher value of this velocity compared to the ones measured presented by Jörg Schlaich et al. (Schlaich J. , Bergermann, Schiel, & Weinrebe, 2004), shown on figure 6, for the prototype built in Spain is expected because the turbine has not been considered in this model. It is expected that the inclusion of the wind turbine in the model will give a closer average values to 9.5 m/s for the time analyzed (June 8th, at noon).

5. CONCLUDING REMARKS

The main considerations for modeling a wide number of energy systems have been presented. Although, the inclusion of additional governing equations for each particular case is needed, the work presented is considered a framework for the modeling of energy generation systems that can be represented by figure 1.

The approach used can also be used to model highly conducting fluids in presence of a magnetic field (Ghung, 2002) or plasma, in which case the inclusion of the governing equations of electromagnetic fields is needed (Park, 2003).

Analytical solutions of the governing equations presented have not been obtained yet, and only solutions by numerical methods are feasible by now. For further details about numerical process to solve systems of equations like those presented on this work, the reader is invited to review publications made by Hughes and collaborators (Hughes, Feijóo, Mazzei, & Quincy, 1998; Hughes, Scovazzi, & Tezduyar, 2008).

The ability to model air flow in natural convection due to solar irradiation is certainly helpful in designing passive solar buildings, thermosyphon collector systems, updraft towers, PV cooling, etc., but the computational burden is considerable.

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